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# A CATEGORIAL ANALYSIS OF LAMBDA CALCULUS MODELS

(Extended Abstract)

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The theory of models for type free lambda calculus was initiated by Dana Scott with the discovery of the $D_\infty$ model. His construction of models consists of two parts: one is that from the category CL of continuous lattices to a reflexive domain, and another is that from the reflexive domain to a $\lambda$-model. Our question which triggered our research is the following:

Is it essential to use partial order relations or some topological properties in the second part of Scott's construction?

In this paper we investigate this problem and have the following two results.

(1) Every $\lambda$-model is the induced groupoid of some reflection of a cartesian closed category.

(2) The induced groupoid of any reflection of $\xi$-extensional cartesian closed categories can be made into a $\lambda$-model.

The first statement says that for every $\lambda$-model (even for a graph model), there is a categorial characterization similar to that of the $D_\infty$ model. On the other hand, the second statement gives a sufficient condition for making the induced groupoid of the given reflection into a $\lambda$-model. So, we can solve the above problem negatively.

The readers may refer to [Bar81] and [Mac71] for the notions that are used in this paper without definitions.

## Extensionality

In this chapter a characterization of the condition of weak extensionality is given in terms of the concepts of extensional subsets.

DEFINITION 1.1. Let $\mathfrak{M} = (X, .)$ be a groupoid and $S \subset X$. Then S is called <u>extensional</u> if

$\forall a, b \in S\ (\forall c \in X.\ ac = bc\ ) \rightarrow a = b.$

DEFINITION 1.2. For a p$\lambda$A $\mathfrak{M} = (X, ., \lambda^*)$,

$F_{\mathfrak{M}} = \{\ (\lambda^* x.\ A)_\rho \mid A \in \mathcal{T}(\mathfrak{M}),\ x \in \mathrm{Vars},\ \rho \in \mathrm{Vals}\ \}.$

THEOREM 1.3. Let $\mathfrak{M}$ be a p$\lambda$A. Then $\mathfrak{M}$ is weakly extensional iff $F_{\mathfrak{M}}$ is extensional.

## From Lambda Models to Cartesian Closed Categories

### 2.1. Retracts of $\lambda$-models

In this section we introduce the notion of retracts of $\lambda$-models and prove that the set of all retracts forms a cartesian closed category (c. c. c.).

Let $\mathfrak{M} = (X, ., \lambda^*)$ be a fixed $\lambda$-model throughout this chapter, and we shall write F instead of $F_{\mathfrak{M}}$.

PROPOSITION 2.1. F is extensional.

DEFINITION 2.2. For $a, b \in X$,

(i) $a \circ b = (\lambda^* x.\ c_a(c_b x))_\rho$,

(ii) $a \rightarrow b = (\lambda^* xy.\ c_b(x(c_a y)))_\rho$,

(iii) $i = (\lambda^* x.\ x)_\rho$.

DEFINITION 2.3. (i) An element r of X is called a <u>retract</u> if $r \circ r = r$.

(ii) $\mathrm{Ret} = \{\ r \in X \mid r \text{ is a retract.}\ \}$.

In this chapter, $r$, $r_1$, $r_2$, ... denote arbitrary retracts.

DEFINITION 2.4. (i) An element a of X is called to have a <u>type</u> $r$, notation $a : r$, if $a = ra$.

(ii) $RET(r_1, r_2) = \{ \langle a, r_1, r_2 \rangle \mid a : r_1 \to r_2 \}$.

(iii) $\mathcal{A} = \bigcup_{(r_1, r_2) \in Ret \times Ret} RET(r_1, r_2)$.

(iv) $\circ'$ is a partial binary operator on $\mathcal{A}$ such that

(1) $\langle b, r_3, r_4 \rangle \circ' \langle a, r_1, r_2 \rangle$ is defined iff $r_3 = r_2$,

(2) $\langle b, r_2, r_3 \rangle \circ' \langle a, r_1, r_2 \rangle = \langle b \circ a, r_1, r_3 \rangle$.

(v) i is a function from Ret to $\mathcal{A}$ such that $i(r) = \langle r, r, r \rangle$.

(vi) $RET = (Ret, \mathcal{A}, \circ', i)$.

THEOREM 2.5. The structure RET is a category.

In the rest of this chapter we shall write a and $b \circ a$ instead of $\langle a, r_1, r_2 \rangle$ and $\langle b, r_2, r_3 \rangle \circ' \langle a, r_1, r_2 \rangle$, respectively if there occurs no confusion.

DEFINITION 2.6. (i) $1 = (\lambda^* x.\ i)\rho$.

(ii) $!_r = \langle 1, r, 1 \rangle$.

(iii) $T \equiv \lambda^* xy.\ x$.

(iv) $F \equiv \lambda^* xy.\ y$.

(v) $a \times b = (\lambda^* xy.\ y(c_a(xT))(c_b(xF)))\rho$.

(vi) $p_{ab} = (\lambda^* x.\ c_a(xT))\rho$.

(vii) $q_{ab} = (\lambda^* x.\ c_b(xF))\rho$.

(viii) $\langle a, b \rangle = (\lambda^* xy.\ y(c_a x)(c_b x))\rho$.

(ix) $e_{ab} = (\lambda^* x.\ c_b(xT(c_a(xF))))\rho$.

(x) $a^+ = (\lambda^* xy.\ c_a(\lambda^* z.\ zxy))\rho$.

We can show that 1, $(r_1 \times r_2, p_{r_1 r_2}, q_{r_1 r_2})$ and $(r_1 \to r_2, e_{r_1 r_2})$ are a terminal, a product and an exponentiation of two retracts $r_1$ and $r_2$, respectively.

THEOREM 2.7. The structure

$RET = (Ret, \mathcal{A}, \circ', i, 1, !, \times, p, q, \langle\ \rangle, \to, e, ^+)$ is a c. c. c.

2.2. A Reflection of the Category of Retracts

In this section we show that the groupoid (X, .) can be seemed as a induced groupoid of some reflection of the c. c. c. RET.

First, we define the notions of reflections and their induced groupoids.

DEFINITION 2.8. Let the structure

$\mathcal{C} = (\mathcal{O}, \mathcal{A}, \circ, i, 1, !, \times, p, q, \langle\ \rangle, \rightarrow, e, {}^{+})$ be a c. c. c.

(i) For $a \in \mathcal{C}$, $\tilde{a} = \mathcal{C}(1, a)$.

(ii) A triple (r, f, g) with $r \in \mathcal{C}$, $f \in \mathcal{C}(r \rightarrow r, r)$ and $g \in \mathcal{C}(r, r \rightarrow r)$ is called a <u>reflection</u> if

(1) $\mathrm{Card}(\tilde{r}) > 1$,

(2) $g \circ f = i_{r \rightarrow r}$.

(iii) The <u>induced</u> <u>groupoid</u> of a reflection (r, f, g) is a groupoid $(\tilde{r}, *)$, where $a * b = e_{rr}\langle ga, b\rangle$ for each a and b in $\tilde{r}$.

PROPOSITION 2.9. A triple $(i, i \rightarrow i, i \rightarrow i)$ is a reflection.

DEFINITION 2.10. The function $\varphi : \tilde{i} \rightarrow X$ is defined by $\varphi(a) = ai$ for each $a \in \tilde{i}$.

THEOREM 2.11. The function $\varphi$ is an isomorphism.

Identifying all the isomorphic structures according to the custom in algebra, we can sum up the results of this chapter in the following.

COROLLARY 2.12. For a groupoid $\mathfrak{M}$, if $\mathfrak{M}$ can be made into a $\lambda$-model, then $\mathfrak{M}$ is the induced groupoid of some reflection of a c. c. c.

The converse of this corollary will be investigated in the next chapter.

From Cartesian Closed Categories to Lambda Models

3.1. $\gamma$-theories and Cartesian Closed Categories

In this section we introduce the equational theory $\gamma$ as a tool of studying syntactic aspects of c. c. c.

DEFINITION 3.1. (i) A class of primitive types P is a non-trivial class with an initial type $0 \in P$.

(ii) The class of types over a class of primitive types P, notation $Typ_P$ or Typ, is a class inductively defined by

(1) $P \subset Typ_P$,

(2) $t_1, t_2 \in Typ_P \Rightarrow (t_1 \times t_2) \in Typ_P$,

(3) $t_1, t_2 \in Typ_P \Rightarrow (t_1 \to t_2) \in Typ_P$.

In this section t, $t_1$,... denote arbitrary types in $Typ_P$.

DEFINITION 3.2. (i) For each $t \in Typ$, $Vars_t$ is a given countable set such that $t_1 = t_2 \Rightarrow Vars_{t_1} \cap Vars_{t_2} = 0$.

(ii) $Vars = \bigcup_{t \in Typ} Vars$ .

DEFINITION 3.3. (i) The class of $\gamma$-terms over P, notation $\Gamma_P$ or $\Gamma$, is a class expressed as a disjoint union of sets of the form $\Gamma_P = \bigcup\{\Gamma_P(t_1, t_2) \mid t_1, t_2 \in Typ_P\}$, where $\Gamma_P(t_1, t_2)$ is the family of minimum sets satisfying the following nine conditions;

(1) $Vars_t \subset \Gamma_P(0, t)$, (2) $I_t \in \Gamma_P(t, t)$, (3) $O_t \in \Gamma_P(t, 0)$,

(4) $P_{t_1 t_2} \in \Gamma_P(t_1 \times t_2, t_1)$, (5) $Q_{t_1 t_2} \in \Gamma_P(t_1 \times t_2, t_2)$,

(6) $E_{t_1 t_2} \in \Gamma_P((t_1 \to t_2) \times t_1, t_2)$,

(7) $A \in \Gamma_P(t_1, t_2)$ and $B \in \Gamma_P(t_2, t_3) \Rightarrow (BA) \in \Gamma_P(t_1, t_3)$,

(8) $A \in \Gamma_P(t_1, t_2)$ and $B \in \Gamma_P(t_1, t_3)$

$$\Rightarrow \langle A, B\rangle \in \Gamma_P(t_1, t_2 \times t_3),$$

(9) $A \in \Gamma_P(t_1 \times t_2, t_3)$ $A^+ \in \Gamma_P(t_1, t_2 \to t_3)$.

(ii) $I_t$, $O_t$, $P_{t_1 t_2}$, $Q_{t_1 t_2}$ and $E_{t_1 t_2}$ are called constant symbols.

(iii) Let $A, B \in \Gamma_P(t_1, t_2)$. Then a form $A = B$ is called a $\gamma$-formula.

(iv) Let $A$ and $B$ be $\gamma$-terms. Then a notation $A \equiv B$ denotes syntactic equality.

$A, B, \ldots$ denote arbitrary $\gamma$-terms, and all the subscripts which does not cause any ambiguity will be omitted.

DEFINITION 3.4. (i) The formal theory $\gamma$ over P is an equational theory on $\Gamma_P$ whose axiom schemes and deduction rules are the following;

(1) $A = A$, (2) $(AB)C = A(BC)$,

(3) $IA = A$, (4) $AI = A$,

(5) $A = O_t$ for $A \in \Gamma(t, O)$, (6) $P\langle A, B\rangle = A$,

(7) $Q\langle A, B\rangle = B$, (8) $\langle PA, QA\rangle = A$,

(9) $E\langle A^+P, Q\rangle = A$, (10) $(E\langle AP, Q\rangle)^+ = A$,

(11) $\dfrac{A = B}{B = A}$, (12) $\dfrac{A = B,\quad B = C}{A = C}$,

(13) $\dfrac{A = C,\quad B = D}{AB = CD}$, (14) $\dfrac{A = C,\quad B = D}{\langle A, B\rangle = \langle C, D\rangle}$,

(15) $\dfrac{A = B}{A^+ = B^+}$.

(ii) A notation $\gamma \vdash A = B$ is defined as usual.

(iii) For $A \in \Gamma$, $FV(A) \subset Vars$ is the set of all variables occured in A.

(iv) For $A \in \Gamma$, $x \in Vars$ and $B \in \Gamma(O, t)$, $A[x := B]$ is the $\gamma$-term obtained by substituting all the x in A by B.

DEFINITION 3.5. (i) For $x \in Vars_{t_1}$ and $A \in \Gamma(t_3, t_2)$, $\mathbf{k}x.\ A \in \Gamma(t_3 \times t_1, t_2)$ is defined by

(1) $\mathbf{k}x.\ x \equiv Q$,

(2) $\kappa x.\ y \equiv yP$ if $x \not\equiv y \in Vars$,

(3) $\kappa x.\ C \equiv CP$ if $C$ is a constant symbol,

(4) $\kappa x.\ AB \equiv (\kappa x.\ A)\langle \kappa x.\ B,\ Q\rangle$,

(5) $\kappa x.\ \langle A,\ B\rangle \equiv \langle \kappa x.\ A,\ \kappa x.\ B\rangle$,

(6) $\kappa x.\ A^{+} \equiv ((\kappa x.\ A)\langle\langle PP,\ Q\rangle,\ QP\rangle)^{+}$.

(ii) For $x \in Vars_{t_1}$ and $A \in \Gamma(t_3,\ t_2)$, $\lambda x.\ A \in \Gamma(t_3,\ t_1 \to t_2)$ is defined by $\lambda x.\ A \equiv (\kappa x.\ A)^{+}$.

The following theorem is called the functional completeness theorem for the theory $\gamma$.

THEOREM 3.6. For $A \in \Gamma(t_3,\ t_2)$, $x \in Vars_{t_1}$ and $B \in \Gamma(0,\ t_1)$,

(i) $FV(\lambda x.\ A) = FV(A) - \{\ x\ \}$.

(ii) $\gamma \vdash E_{t_1 t_2}\langle \lambda x.\ A,\ BO_{t_3}\rangle = A[x := B]$.

Let $\mathcal{C} = (\mathcal{O},\ \mathcal{A},\ \circ,\ i,\ 1,\ !,\ \times,\ p,\ q,\ \langle\ \rangle,\ \to,\ e,\ ^{+})$ be a fixed c. c. c. in the rest of this chapter.

DEFINITION 3.7. (i) $P = \{\ t\ \} \times \mathcal{O}$.

(ii) We write $t_a$ instead of $(t,\ a)$.

(iii) $0 \equiv t_1$ (initial type).

PROPOSITION 3.8. $P$ is a class of primitive types.

DEFINITION 3.9. (i) $Typ_{\mathcal{C}} = Typ_{P}$.

(ii) For $t \in Typ_{\mathcal{C}}$, $\bar{t} \in \mathcal{C}$ is defined by

(1) $\overline{t_a} = a$, (2) $\overline{t_1 \times t_2} = \overline{t_1} \times \overline{t_2}$, (3) $\overline{t_1 \to t_2} = \overline{t_1} \to \overline{t_2}$.

DEFINITION 3.10. (i) The class of <u>extended</u> <u>$\gamma$-terms</u> over $\mathcal{C}$, notation $\Gamma_{\mathcal{C}}$, is a class expressed as a disjoint union of sets of the form $\Gamma_{\mathcal{C}} = \cup\{\ \Gamma_{\mathcal{C}}(t_1,\ t_2)\ \mid\ t_1,\ t_2 \in Typ_{\mathcal{C}}\ \}$, where $\Gamma_{\mathcal{C}}(t_1,\ t_2)$ is the family of minimum sets satisfying the following;

(1) $Vars_t \subset \Gamma_{\mathcal{C}}(0,\ t)$ (variables),

(2) $f \in \mathcal{C}(\overline{t_1},\ \overline{t_2}) \Rightarrow c_f \in \Gamma_{\mathcal{C}}(t_1,\ t_2)$ (constant symbols),

(3) $A \in \Gamma_{\mathcal{C}}(t_1, t_2)$ and $B \in \Gamma_{\mathcal{C}}(t_2, t_3) \Rightarrow (BA) \in \Gamma_{\mathcal{C}}(t_1, t_3)$,

(4) $A \in \Gamma_{\mathcal{C}}(t_1, t_2)$ and $B \in \Gamma_{\mathcal{C}}(t_1, t_3)$

$$\Rightarrow \langle A, B\rangle \in \Gamma_{\mathcal{C}}(t_1, t_2 \times t_3),$$

(5) $A \in \Gamma_{\mathcal{C}}(t_1 \times t_2, t_3) \Rightarrow A^+ \in \Gamma_{\mathcal{C}}(t_1, t_2 \to t_3)$.

(ii) $I_t$, $O_t$, $P_{t_1 t_2}$, $Q_{t_1 t_2}$ and $E_{t_1 t_2}$ are names of $i_{\bar{t}}$, $!_{\bar{t}}$, $p_{\bar{t}_1 \bar{t}_2}$, $q_{\bar{t}_1 \bar{t}_2}$ and $e_{\bar{t}_1 \bar{t}_2}$, respectively.

DEFINITION 3.11. (i) A function $\rho$ : Vars $\to \mathcal{A}$ is called a <u>valuation</u> in $\mathcal{C}$ if it satisfies $x \in \mathrm{Vars}_t \Rightarrow \rho(x) \in \mathcal{C}(1, \bar{t})$.

(ii) For $A \in \Gamma_{\mathcal{C}}$ and a valuation $\rho$, the <u>interpretation</u> of A in $\mathcal{C}$ under $\rho$, notation $[A]^{\mathcal{C}}_{\rho}$ or $[A]_{\rho}$, is defined by

(1) $[x]^{\mathcal{C}}_{\rho} = \rho(x)$ if $x \in$ Vars, (2) $[c_f]^{\mathcal{C}}_{\rho} = f$,

(3) $[AB]^{\mathcal{C}}_{\rho} = [A]^{\mathcal{C}}_{\rho}[B]^{\mathcal{C}}_{\rho}$, (4) $[\langle A, B\rangle]^{\mathcal{C}}_{\rho} = \langle [A]^{\mathcal{C}}_{\rho}, [B]^{\mathcal{C}}_{\rho}\rangle$,

(5) $[A^+]^{\mathcal{C}}_{\rho} = ([A]^{\mathcal{C}}_{\rho})^+$.

DEFINITION 3.12. Let $A, B \in \Gamma_{\mathcal{C}}$ and $\rho$ be a valuation.

(i) $\mathcal{C}, \rho \models A = B$ iff $[A]^{\mathcal{C}}_{\rho} = [B]^{\mathcal{C}}_{\rho}$.

(ii) $\mathcal{C} \models A = B$ iff $\mathcal{C}, \rho \models A = B$ for every valuation $\rho$.

DEFINITION 3.13. The <u>extended</u> <u>$\gamma$-theory</u> over $\mathcal{C}$, notation $\gamma(\mathcal{C})$, is the extension of the theory $\gamma$ obtained by validating the axiomschemas and rules also for terms in $\Gamma_{\mathcal{C}}$.

The following is the key theorem to understand the relation between $\gamma$-theories and c. c. c.

THEOREM 3.14. For extended $\gamma$-terms A and B,

$\gamma(\mathcal{C}) \vdash A = B \Rightarrow \mathcal{C} \models A = B$.

DEFINITION 3.15. Notations $FV(A)$, $A[x := B]$, $\kappa x.\ A$ and $\lambda x.\ A$ are extended for terms in $\Gamma_{\mathcal{C}}$ reasonably.

Theorem 3.6 remains valid for the theory $\gamma(\mathcal{C})$.

THEOREM 3.16. For $A \in \Gamma_{\mathcal{C}}(t_3, t_2)$, $x \in \mathrm{Vars}_{t_1}$ and $B \in \Gamma_{\mathcal{C}}(O, t_1)$, $\mathcal{C} \models E_{t_1 t_2}\langle \lambda x.\ A, BO_{t_3}\rangle = A[x := B]$.

This theorem is a generalization of the result of Lambek

[Lam74].

DEFINITION 3.17. $\mathcal{C}$ is $\xi$-extensional if for all A, B $\in \Gamma_{\mathcal{C}}$ and $x \in$ Vars, $\mathcal{C} \models A = B \Rightarrow \mathcal{C} \models \lambda x.\ A = \lambda x.\ B$.

3.2. Reflections of Cartesian Closed Categories

In this section we show that the induced groupoid of any reflection of c. c. c. can be made into a p$\lambda$A.

Let (r, f, g) be a fixed reflection of $\mathcal{C}$, and ($\tilde{r}$, $*$) be the induced groupoid of (r, f, g).

DEFINITION 3.18. (i) $t \equiv t_r$.

(ii) $\mathcal{R} = \Gamma_{\mathcal{C}}(0, t)$.

(iii) $F \equiv c_f$.

(iv) $G \equiv c_g$.

(v) For A, B $\in \mathcal{R}$, $A * B \equiv E_{tt}\langle GA, B\rangle$.

In this section A, B,... denote arbitrary extended $\gamma$-terms in $\mathcal{R}$, and x, y,... denote arbitrary variables in $\mathrm{Vars}_t$.

DEFINITION 3.19. (i) $\lambda^\circ x.\ A \equiv F(\lambda x.\ A)$.

(ii) $\lambda^\circ x_1 \ldots x_n.\ A \equiv \lambda^\circ x_1.\ (\lambda^\circ x_2.\ (\ldots(\lambda^\circ x_n.\ A)\ldots))$.

(iii) $k = [\lambda^\circ xy.\ x]_\rho$.

(iv) $s = [\lambda^\circ xyz.\ x * z * (y * z)]_\rho$.

(v) For $A \in \mathcal{T}(\tilde{r}, *)$ and $x \in \mathrm{Vars}_t$, $\lambda' x.\ A \in \mathcal{T}(\tilde{r}, *)$ is defined by

(1) $\lambda' x.\ x \equiv c_{skk}$, (2) $\lambda' x.\ y \equiv c_k y$ if $y \not\equiv x$, .

(3) $\lambda' x.\ c_f \equiv c_k c_f$, (4) $\lambda' x.\ AB \equiv c_s(\lambda' x.\ A)(\lambda' x.\ B)$.

In the rest of this section, $\mathfrak{M}$ denotes ($\tilde{r}$, $*$, $\lambda'$).

THEOREM 3.20. $\mathfrak{M}$ is a p$\lambda$A.

For making $\mathfrak{M}$ into a $\lambda$-model, it is sufficient that $F_{\mathfrak{M}}$ is extensional by 1.3. We shall prove that the condition of $\xi$-extensionality is sufficient to make $F_{\mathfrak{M}}$ extensional in the next theorem.

THEOREM 3.21. The induced groupoid of any reflection of $\xi$-extensional c. c. c. can be made into a $\lambda$-model.

This theorem is a partial result concerning with the converse of 2.12.

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